§7.9.

proposition 7.9.

a), b), e) have been proved.

e) The set of simple roots Π are these:

Case 1: $\alpha_1 = \alpha_1'$, $\cdots$, $\alpha_{\ell-1} = \alpha_{\ell-1}'$, $\alpha_\ell = \frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}')$

Case 2: $\alpha_1 = \frac{1}{2}(\alpha_1' + \alpha_{2\ell-1}')$, $\cdots$, $\alpha_\ell = \frac{1}{2}(\alpha_{\ell-1}' + \alpha_{\ell+1}')$, $\alpha_\ell = \alpha_\ell'$

Case 3: $\alpha_1 = \alpha_6'$, $\alpha_2 = \alpha_3'$, $\alpha_3 = \frac{1}{2}(\alpha_2' + \alpha_4')$, $\alpha_4 = \frac{1}{2}(\alpha_1' + \alpha_5')$

Case 4: $\alpha_1 = \alpha_2'$, $\alpha_2 = \frac{1}{3}(\alpha_1' + \alpha_3' + \alpha_4')$

c) The normalized bilinear form $(\cdot|\cdot)$ on g is given by the following formulas: $(\alpha|\alpha) = 2$ if $\alpha \in \Delta_\ell$.

(1) $(h|h') = (h|h')'|_H$, $(h|E_\alpha) = 0$ if $h, h' \in H$, $\alpha \in \Delta$

(2) $(E_\alpha|E_\beta) = -\delta_{\alpha,-\beta}$ (resp. $-r\delta_{\alpha,-\beta}$) if $\alpha, \beta \in \Delta_\ell$ (resp $\in \Delta_s$).
$(E_\alpha|E_\beta) = 0$ if $\alpha \in \Delta_s$, $\beta \in \Delta_\ell$. $\square$

proof: ① For Case 1 : $X = D_{\ell+1}$.

- $(\alpha_1|\alpha_1) = (\alpha_1'|\alpha_1') = 2$, $\cdots$, $(\alpha_\ell|\alpha_\ell) = (\frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}')|\frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}')) = \frac{1}{2}(\alpha_\ell'|\alpha_\ell')$

$$\alpha_\ell^\vee = \frac{2\nu^{-1}(\alpha_\ell)}{(\alpha_\ell|\alpha_\ell)} = \frac{4(\alpha_\ell^\vee + \alpha_{\ell+1}^\vee)}{(\alpha_\ell' + \alpha_{\ell+1}'|\alpha_\ell' + \alpha_{\ell+1}')} = \underbrace{\alpha_\ell^\vee + \alpha_{\ell+1}^\vee}_{=1}$$

- $\alpha_j^\vee = \alpha_j'^\vee$ $(1 \le j \le \ell-1)$, $\alpha_\ell^\vee = \alpha_\ell'^\vee + \alpha_{\ell+1}'^\vee$
$(\alpha_j^\vee|\alpha_j^\vee) = (\alpha_j'^\vee|\alpha_j'^\vee) = 2$, $(1 \le j \le \ell-1)$, $(\alpha_\ell^\vee|\alpha_\ell^\vee) = 4$.

② For case 2: $X = A_{2\ell-1}$.
$(\alpha_1|\alpha_1) = (\frac{1}{2}(\alpha_1' + \alpha_{2\ell-1}')|\frac{1}{2}(\alpha_1' + \alpha_{2\ell-1}')) = 1$, $\cdots$, $(\alpha_\ell|\alpha_\ell) = (\alpha_\ell'|\alpha_\ell') = 2$.

③ For cas3: $X = E_6$.
$(\alpha_1|\alpha_1) = (\alpha_2|\alpha_2) = 2$, $(\alpha_3|\alpha_3) = (\alpha_4|\alpha_4) = 1$. ✓.

④ For case 4: $X = D_4$.
$(\alpha_1|\alpha_1) = 2$, $(\alpha_2|\alpha_2) = \frac{2}{3}$ ✓.

(2). if $\alpha, \beta \in \Delta_\ell$. Then $(E_\alpha|E_\beta) = (E_\alpha'|E_\beta')' = -\delta_{\alpha,-\beta}$.
if $\alpha, \beta \in \Delta_s$, Then $(E_\alpha|E_\beta) = (E'_{\bar\mu(\alpha')} + \cdots + E'_{\bar\mu^r(\alpha')}|E'_{\bar\mu(\beta')} + \cdots + E'_{\bar\mu^r(\beta')})$.
$(r=2)$ $= (E'_{\bar\mu(\alpha')}|E'_{\bar\mu(\beta')})' + (E'_{\bar\mu(\alpha')}|E'_{\beta'})' + (E'_{\alpha'}|E'_{\bar\mu(\beta')})' + (E'_{\alpha'}|E'_{\beta'})$

[$\alpha = \frac{\bar\mu(\alpha') + \alpha'}{2}$, $\beta = \frac{\bar\mu(\beta') + \beta'}{2}$, if $\alpha = -\beta$, i.e. $\bar\mu(\alpha') + \alpha' = -\bar\mu(\beta') - \beta'$.
Then we have $\alpha' = -\beta'$ or $\alpha' = -\bar\mu(\beta')$]

- if $\alpha' = -\beta'$, 左式 $= (E'_{\bar\mu(\alpha')}|E'_{\bar\mu(\beta')})' + (E'_{\alpha'}|E'_{\beta'})' = -1-1 = 2$
The same as $\alpha' = -\bar\mu(\beta')$.

- if $\alpha \ne -\beta$, $\Rightarrow \alpha' \ne -\beta'$ & $\alpha' \ne -\bar\mu(\beta')$ 左式 $= 0$.

Then we have $(E_\alpha|E_\beta) = -r\delta_{\alpha,-\beta}$.

d). Δ is the set of roots of g with respect to the Cartan subalgebra H, Δ_s (resp. Δ_ℓ) be the set of short (resp. long) roots and Q is its root lattice, $\Delta^\vee = \Delta_\ell \cup r\Delta_s$.

proof: if $\alpha \in \Delta_\ell$, then $\alpha^\vee = \alpha$.
if $\alpha \in \Delta_s$, then $\alpha^\vee = \frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)} = \frac{2(\frac{1}{r}\nu^{-1}(\bar\mu(\alpha')+\cdots+\bar\mu^r(\alpha')))}{\frac{1}{r^2}(\bar\mu(\alpha')+\cdots\bar\mu^r(\alpha')|\bar\mu(\alpha')+\cdots+\bar\mu^r(\alpha'))}$

$= \frac{2r(\bar\mu(\alpha')^\vee+\cdots\bar\mu^r(\alpha')^\vee)}{2r} = \bar\mu(\alpha)^\vee+\cdots+\bar\mu^r(\alpha')^\vee$

$= r\alpha$.

f) Letting $\Pi_\ell = \Pi\cap\Delta_\ell$, $\Pi_s = \Pi\cap\Delta_s$, we have $\Pi^\vee = \Pi_\ell \cup r\Pi_s$.

g). Case 1: $\theta_0 = \alpha_1+\alpha_2+\cdots+\alpha_\ell$. ✓

Case 2: $\theta_0 = \alpha_1+2\alpha_2+\cdots+2\alpha_{\ell-1}+\alpha_\ell$ ✓

Case 3: $\theta_0 = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4$

Case 4: $\theta_0 = \alpha_1+2\alpha_2$.

$g^{(j)}$.

proof: $[g, g^{(j)}] = g^{(j)}$ Rem 7.9 b).
Since $[h^{(0)}, E_\alpha^{(j)}] = (h^{(0)}|\alpha)E_\alpha^{(j)}$ where $h^{(0)}\in\mathcal{H}$, $\alpha\in\Delta$.

We want to find the highest weight α, It is obviously to check the highest root on g. Equivalently to find the highest j-element orbit.

· Case 1: $\Delta(D_{\ell+1}) = \pm\{\alpha_i'+\alpha_{i+1}'+\cdots+\alpha_{j-1}'+2\alpha_j'+2\alpha_{j+1}+\cdots+2\alpha_{\ell-1}'+\alpha_\ell'+\alpha_{\ell+1}'$
$(1\le i<j<\ell)$ & $\alpha_i'+\alpha_{i+1}'+\cdots+\alpha_j'$, $1\le i<\ell$, & $i<j\le\ell+1\}$

The highest 2-element orbit:
$(\underbrace{(\alpha_1'+\cdots+\alpha_{\ell-1}')+\alpha_\ell'}_{A} \xrightarrow{\bar\mu} \underbrace{(\alpha_1'+\cdots+\alpha_{\ell-1}')+\alpha_{\ell+1}'}_{B})$.

then the highest weight $\alpha = \frac{A+B}{2} = \alpha_1+\cdots+\alpha_{\ell-1}+\alpha_\ell$.
$[h^{(0)}|E_\alpha^{(j)}] = (h|\underbrace{\frac{\alpha'+\bar\mu(\alpha')}{2}})'E_\alpha^{(j)}$

· Case 2: $\Delta(A_{2\ell-1}) = \pm\{\alpha_i'+\alpha_{i+1}'+\cdots+\alpha_j' \mid 1\le i\le j\le 2\ell\}$.
The highest 2-element orbit:
$(\underbrace{\alpha_1'+\cdots+\alpha_{2\ell-2}'}_{A} \xrightarrow{\bar\mu} \underbrace{\alpha_2'+\cdots+\alpha_{2\ell-1}'}_{B})$. 唯一,最高.

Then $\alpha = \frac{A+B}{2} = \alpha_1+2\alpha_2+\cdots+2\alpha_{\ell-1}+\alpha_\ell$.

$\underbrace{e_i \quad f_j}$ 从极大半单部分
同步对角化.

Remark 7.9.
(a) (7.8.5) is a special case of (7.9.3) for $r=1$.
· $r=1$ simply-laced, we have $\alpha\pm\beta\in\Delta$ iff $(\alpha|\beta)=\mp1$.
then $\alpha+\beta\in\Delta \Rightarrow \alpha-\beta\notin\Delta \Rightarrow p=0$.

(b). Commutation relations (7.9.3) can be extended to the whole $g(X_N)$ as follows:
Let $E_\alpha^{(0)} = E_\alpha = E_\alpha'$ if $\alpha\in\Delta_\ell$.

$\alpha^{(i)} = \eta^{-i}\bar{\mu}(\alpha') + \cdots + \eta^{-ri}\bar{\mu}^{r}(\alpha')$ if $\alpha \in \Delta_s$

Then $(0 \le i, j \le r-1)$:

$$\begin{cases} ② \ [h^{(i)}, E_\alpha^{(j)}] = (h^{(i)} | \alpha) E_\alpha^{(i+j)}, \text{ if } h^{(i)} \in \mathcal{H}^{(i)}, \alpha \in \Delta. \\ ③ \ [E_\alpha^{(i)}, E_{-\alpha}^{(j)}] = \begin{cases} -\alpha^{(i+j)}, & \text{if } \alpha \in \Delta_s \\ -\alpha, & \text{if } \alpha \in \Delta_l. \end{cases} \end{cases}$$

proof: ② if $i=1$, $j=1$.

$$\begin{aligned}
[h^{(1)}, E_\alpha^{(1)}] &= [h^{(1)}, \eta^{-1} E'_{\bar{\mu}(\alpha')} + \eta^{-2} E'_{\alpha'}] \\
&= (h^{(1)} | \bar{\mu}(\alpha')) \eta^{-1} E'_{\bar{\mu}(\alpha')} + (h^{(1)} | \alpha') \eta^{-2} E'_{\alpha'} \\
&= (\bar{\mu}(h^{(1)}) | \alpha') \eta^{-1} E'_{\bar{\mu}(\alpha')} + (h^{(1)} | \alpha') \eta^{-2} E'_{\alpha'} \\
&= (h^{(1)} | \alpha') (E'_{\bar{\mu}(\alpha')} + \eta^{-2} E'_{\alpha'}) = (h^{(1)} | \alpha') (\eta^{-2} E'_{\bar{\mu}(\alpha')} + \eta^{-4} E'_{\alpha'}) \\
&= (h^{(1)} | \alpha) E_\alpha^{(2)}.
\end{aligned}$$

③ if $i=j=1$ & $\Delta \in \Delta_s$.

$$\begin{aligned}
[E_\alpha^{(1)}, E_{-\alpha}^{(1)}] &= [\eta^{-1} E'_{\bar{\mu}(\alpha')} + E'_{\alpha'}, \eta^{-1} E'_{-\bar{\mu}(\alpha')} + E'_{-\alpha'}] \\
&= \eta^{-2} [E'_{\bar{\mu}(\alpha')}, E'_{-\bar{\mu}(\alpha')}] + \eta^{-1} [E'_{\bar{\mu}(\alpha')}, E'_{-\alpha'}] + \eta^{-1} [E'_{\alpha'}, E'_{-\bar{\mu}(\alpha')}] \\
&\quad + [E'_{\alpha'}, E'_{-\alpha'}] \\
&= \eta^{-2}(-\bar{\mu}(\alpha')) - \alpha' = \eta^{-4}(-\bar{\mu}(\alpha')) - \eta^{-2}\alpha' \\
&= -\alpha^{(2)}.
\end{aligned}$$

(c) The chevally involution ω and the compact involution ω_0 of $g(X_N)$ are given by $(\alpha \in \Delta)$

$\omega(E_\alpha) = E_{-\alpha}$, $\omega(\alpha) = -\alpha$, $\omega_0(E_\alpha) = E_{-\alpha}$, $\omega_0(\alpha) = -\alpha$.

(d) The Hermitian form $(\cdot|\cdot)_0$ is positive-definite on $g(X_l)$, where X_l is of finite type.

proof: we know: $(\cdot|\cdot)_0 = -(\omega_0(x)|y)$.

§7.10.

Case 5: $Q(A_{2l})$: $\bar{\mu}(v_i) = -v_{2l+2-i}$

$\Delta(A_{2l}) = \{ v_i - v_j \mid 1 \le i, j \le 2l, i \ne j \}$.

$\Pi = \{ \alpha_1 = v_1 - v_2, \alpha_2 = v_2 - v_3, \cdots, \alpha_{2l} = v_{2l} - v_{2l+1} \}$.

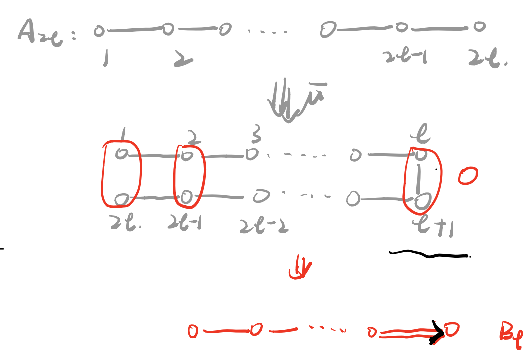
Then $\bar{\mu}(\alpha_i) = \alpha_{2l+1-i}$

In this case. There is no $\bar{\mu}$-invariant orientation, we consider the orientation

$\circ \to \circ \to \cdots \to \circ$.

Then the diagram automorphism μ of $g(A_{2l})$ is defined by:

$\mu(\alpha) = \bar{\mu}(\alpha)$, $\mu(E'_\alpha) = (-1)^{1+ht\alpha} E'_{\bar{\mu}(\alpha)}$



Let $\Delta_\ell = \{ \frac{1}{2}(\alpha' + \bar{\mu}(\alpha')) \mid \alpha' \neq \bar{\mu}(\alpha') \text{ and } (\alpha' \mid \bar{\mu}(\alpha'))' = 0 \}$

$\Delta_s = \{ \frac{1}{2}(\alpha' + \bar{\mu}(\alpha')) \mid \alpha' \neq \bar{\mu}(\alpha') \text{ and } (\alpha' \mid \bar{\mu}(\alpha'))' \neq 0 \}$.

$\Delta = \Delta_\ell \cup \Delta_s$.

$$E_\alpha = \begin{cases} E'_{\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')}, & \text{if } \alpha \in \Delta_\ell. \\ \sqrt{2}\,(E'_{\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')}), & \text{if } \alpha \in \Delta_s. \end{cases}$$

$\mathfrak{H} = \{ h \in \mathfrak{H}' \mid \mu(h) = h \}$. $\mathfrak{H}^{(1)} = \{ h \in \mathfrak{H}' \mid \mu(h) = -h \}$

$\mathfrak{g} = \mathfrak{H} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{C} E_\alpha)$

$\mathfrak{g}^{(1)} = \mathfrak{H}^{(1)} \oplus (\sum_{\substack{\alpha \in \Delta' \\ \alpha \neq \mu(\alpha)}} \mathbb{C}(E'_\alpha + (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha)})) \oplus (\sum_{\substack{\alpha \in \Delta' \\ \alpha = \mu(\alpha)}} \mathbb{C} E'_\alpha)$

prop 7.10.

a) $\mathfrak{g}(A_{2\ell}) = \mathfrak{g} \oplus \mathfrak{g}^{(1)}$

proof: $\mathfrak{g}$ is the eigenspace of μ associatives to the eigenvalue 1.

· if $\alpha \in \Delta_\ell$. $\mu(E_\alpha) = \mu(E'_{\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')})$

$= (-1)^{1+ht\,\alpha'} E'_{\bar{\mu}(\alpha')} - (-1)^{ht\,\alpha} (-1)^{1+ht\,\bar{\mu}(\alpha')} E'_{\alpha'}$

$= E'_{\alpha'} + (-1)^{1+ht\,\alpha'} E'_{\bar{\mu}(\alpha')}$

$= E'_{\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')} = E_\alpha$.

$\underbrace{ht(\alpha)}_{\text{in } \mathfrak{g}} = \underbrace{ht(\alpha') = ht\,\bar{\mu}(\alpha')}_{\text{in } \mathfrak{g}(A_{2\ell})}$.

$\frac{1}{2}(\quad)$ $\mathfrak{g}(A_{2\ell})$.

· if $\alpha \in \Delta_s$. $\mu(E_\alpha) = E_\alpha$.

& $\mu(h) = h$ where $h \in \mathfrak{H}$.

· $\mathfrak{g}^{(1)}$ is the eigenspace of μ associatives to the eigenvalue -1.

From the construction, we have $\mu(h') = -h$, if $h' \in \mathfrak{H}^{(1)}$.

· if $\alpha = \mu(\alpha)$, $\alpha \in \Delta'$. Then $\mu(E'_\alpha) = (-1)^{1+ht\,\alpha} E'_{\bar{\mu}(\alpha)} = -E'_\alpha$.

实际这些根一定包含 $\alpha_\ell + \alpha_{\ell+1}$ & $ht\,\alpha = 2s$, $s \in \{1, \dots, \ell\}$.

· if $\alpha \neq \mu(\alpha)$, $\alpha \in \Delta'$.

then $\mu(E'_\alpha + (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha)}) = (-1)^{1+ht\,\alpha} E'_{\bar{\mu}(\alpha)} + (-1)^{ht\,\alpha}(-1)^{1+ht\,\bar{\mu}(\alpha)} E'_\alpha$

$= -(E'_\alpha + (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha)})$.

b) $\mathfrak{g}$ is the simple Lie algebra of type B_ℓ. and its commutation relation are given by (7.9.3).

proof: $\alpha_{\ell-i}|_{\mathfrak{H}} = \alpha_{\ell+i+1}|_{\mathfrak{H}}$ $(0 \le i \le \ell-1)$.

③ $[E_\alpha, E_{-\alpha}] = -2\alpha$ (resp. $-2r\alpha$) if $\alpha \in \Delta_\ell$ (resp. $\alpha \in \Delta_s$).

proof: if $\alpha \in \Delta_\ell$. Then $[E_\alpha, E_{-\alpha}] = [E'_{\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')}, E'_{-\alpha'} - (-1)^{ht\,\alpha} E'_{\bar{\mu}(-\alpha')}]$

$= [E'_{\alpha'}, E'_{-\alpha'}] - [E'_{\alpha'}, (-1)^{ht\,\alpha} E'_{\bar{\mu}(-\alpha')}] - [(-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha')}, E'_{-\alpha'}] + [E'_{\bar{\mu}(\alpha')}, E'_{\bar{\mu}(-\alpha')}]$

$\alpha' - \bar{\mu}(\alpha') \notin \Delta$

$= [E'_{\alpha'}, E'_{-\alpha'}] + [E'_{\bar{\mu}(\alpha')}, E'_{\bar{\mu}(-\alpha')}]$

$= -\alpha' - \bar{\mu}(\alpha') = -2\alpha$. $(\alpha = \frac{1}{2}(\alpha' + \bar{\mu}(\alpha')))$

If $\alpha \in \Delta_s$. $[E_\alpha, E_\alpha] = -4\alpha$.

A: $\alpha_i + \cdots + \alpha_j$ $\alpha_1 + \alpha_2 \xrightarrow{\mu} \alpha_{2\ell} + \alpha_{2\ell-1}$.

e). The set of simple roots Π is that:

$\alpha_1 = \frac{1}{2}(\alpha_1' + \alpha_{2\ell}')$, $\alpha_2 = \frac{1}{2}(\alpha_2' + \alpha_{2\ell-1}')$, ……, $\alpha_\ell = \frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}')$.

c). The normalized bilinear form $(\cdot|\cdot)$ on g is given by:

(1) $(h|h') = 2(h|h')'|_{H}$, $(h|E_\alpha) = 0$ if $h, h' \in H$, $\alpha \in \Delta$.

(2) $(E_\alpha | E_\beta) = \begin{cases} -2\delta_{\alpha,-\beta}, & \text{if } \alpha, \beta \in \Delta_\ell \\ -4\delta_{\alpha,-\beta}, & \text{if } \alpha, \beta \in \Delta_\ell. \end{cases}$

$(E_\alpha | E_\beta) = 0$ if $\alpha \in \Delta_s$, $\beta \in \Delta_\ell$.

proof: (1). $(\alpha_i|\alpha_i)' = (\frac{1}{2}(\alpha_i' + \alpha_{2\ell-i}') | \frac{1}{2}(\alpha_i' + \alpha_{2\ell-i}')) = 1$ $(1 \le i \le \ell-1)$

$(\alpha_\ell|\alpha_\ell)' = (\frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}') | \frac{1}{2}(\alpha_\ell' + \alpha_{\ell+1}')) = \frac{1}{2}$.

$\alpha_i^\vee = \alpha_i'^\vee + \alpha_{2\ell-i+1}'^\vee$ $(1 \le i \le \ell-1)$, $\alpha_\ell^\vee = 2(\alpha_\ell'^\vee + \alpha_{\ell+1}'^\vee)$.

$(\alpha_i^\vee | \alpha_i^\vee)' = (\alpha_i^\vee + \alpha_{2\ell-i+1}^\vee | \alpha_i^\vee + \alpha_{2\ell-i+1}^\vee)' = 4$ $(1 \le i \le \ell-1)$

$(\alpha_\ell^\vee | \alpha_\ell^\vee)' = 8$.

$(h|h') = 2(h|h')'|_H$.

(2). if $\alpha, \beta \in \Delta_\ell$, Then $(E_\alpha | E_\beta) = (E_{\alpha'}' - (-1)^{ht\alpha} E_{\bar\mu(\alpha')}' | E_\beta' - (-1)^{ht\beta} E_{\bar\mu(\beta')}')$

$= (E_\alpha' | E_{\beta'}') - (E_{\alpha'}' | (-1)^{ht\beta} E_{\bar\mu(\beta')}') - (-1)^{ht\alpha}(E_{\bar\mu(\alpha')}' | E_{\beta'}') + (-1)^{ht\alpha + ht\beta}(E_{\bar\mu(\alpha)}' | E_{\bar\mu(\beta')}')$

($\alpha = \frac{\alpha' + \bar\mu(\alpha')}{2}$, $\beta = \frac{\beta' + \bar\mu(\beta')}{2}$ & $(\alpha' | \bar\mu(\alpha'))' = (\beta' | \bar\mu(\beta'))' = 0$

if $\alpha = -\beta$ then we have $\alpha' = -\beta'$ or $\alpha' = -\bar\mu(\beta')$.
if $\alpha \ne -\beta$, then $\alpha' \ne -\beta'$, $\alpha' \ne -\bar\mu(\beta')$.).

if $\alpha' = -\beta'$, Then 上式 $= -1 -1 = -2$,
if $\alpha \ne -\beta$ Then 上式 $= 0$.

if $\alpha, \beta \in \Delta_s$. Then $(E_\alpha | E_\beta) = -4$.

Th. 2.2 P7. e) $[x, y] = (x|y)\nu^{-1}(\alpha)$. for $x \in g_\alpha$, $y \in g_{-\alpha}$, $\alpha \in \Delta$.

if: $\alpha \in \Delta_\ell$. $\alpha = -\beta$. $[E_\alpha, E_{-\alpha}] \overset{?}{=} (E_\alpha | E_{-\alpha})\nu^{-1}(\alpha)$.

LHS $= -2\alpha$. RHS $= -2\nu^{-1}(\frac{1}{2}(\alpha' + \bar\mu(\alpha'))) = -2 \cdot \frac{1}{2}(\alpha'^\vee + \bar\mu^\vee(\alpha'))$

$\alpha = \frac{1}{2}(\alpha' + \bar\mu(\alpha'))$. $= -(\alpha'^\vee + \bar\mu^\vee(\alpha')) = -2\alpha$.

if: $\alpha \in \Delta_s$, $\alpha = -\beta$. $[E_\alpha, E_{-\alpha}] = -4\alpha$.

$(E_\alpha | E_{-\alpha})\nu^{-1}(\alpha) = -4 \cdot \frac{1}{2}(\alpha'^\vee + \bar\mu^\vee(\alpha')) = -4\alpha$.

d) Δ is the set of roots of g with respect to the Cartan subalgebra H, Δ_s (resp Δ_ℓ) be the set of short (resp. long) roots, and Q is its root lattice, $\Delta^\vee = 2\Delta_\ell \cup 4\Delta_s$.

proof: If $\alpha \in \Delta_\ell$, then $\alpha = \frac{1}{2}(\alpha' + \bar{\mu}(\alpha'))$ & $(\alpha' \mid \bar{\mu}(\alpha'))' = 0$.
then $\alpha^\vee = \frac{2\nu^{-1}(\alpha)}{(\alpha\mid\alpha)} = \frac{4(\alpha'^\vee + \bar{\mu}(\alpha')^\vee)}{(\alpha' + \bar{\mu}(\alpha') \mid \alpha' + \bar{\mu}(\alpha'))} = (\alpha'^\vee + \bar{\mu}(\alpha')^\vee) = 2\alpha$.

If $\alpha \in \Delta_s$, then $\alpha = \frac{1}{2}(\alpha' + \bar{\mu}(\alpha'))$ & $(\alpha' \mid \bar{\mu}(\alpha'))' \neq 0$.
then $\alpha^\vee = \frac{4(\alpha'^\vee + \bar{\mu}(\alpha')^\vee)}{2-2+2} = 2(\alpha'^\vee + \bar{\mu}(\alpha')^\vee) = 4\alpha$.

f). Letting $\Pi_\ell = \Pi \cap \Delta_\ell$, $\Pi_s = \Pi \cap \Delta_s$, we have $\Pi^\vee = 2\Pi_\ell \cup 4\Pi_s$.

g). $[g, g^{(1)}] = g^{(1)}$ and the g-module $g^{(1)}$ is irreducible with highest weight $\theta_0 = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell$.

proof: $\cdot\, [g, g^{(1)}] = g^{(1)}$ by Rem. 7.10.

$\cdot\, A_{2\ell}$: $\Delta = \pm\{\alpha_i' + \alpha_{i+1}' + \cdots + \alpha_j' \mid 1 \le i \le j \le 2\ell\}$.

$$g^{(1)} = H^{(1)} \oplus \Big(\sum_{\substack{\alpha\in\Delta' \\ \alpha \neq \mu(\alpha)}} \mathbb{C}(E_\alpha' + (-1)^{ht\,\alpha} E'_{\bar{\mu}(\alpha)})\Big) \oplus \Big(\sum_{\substack{\alpha\in\Delta' \\ \alpha = \mu(\alpha)}} \mathbb{C}E_\alpha'\Big)$$

$\therefore$ The highest weight corresponds to the highest root:

Then $[h, E_{\alpha_1+\cdots+\alpha_{2\ell}}] = (2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_\ell) E_{\alpha_1+\cdots+\alpha_{2\ell}}$.

§7.11 Construction of infinite rank affine algebras.

1. Def: A generalized Cartan matrix of infinite order is called an infinite affine matrix if every one of its principal minors of finite order is positive.

· exercise 4.14: Show that the following is a complete list of connected Dynkin diagrams of generalized Cartan matrices of infinite order s.t. any principal minor of finite order is positive.

A_∞: $\cdots - \circ - \circ - \circ - \cdots$

$A_{+\infty}$: $\circ - \circ - \circ \cdots$

B_∞: $\circ \Leftarrow \circ - \circ - \circ \cdots$

C_∞: $\circ \Rightarrow \circ - \circ \cdots$

D_∞: $\circ - \overset{\circ}{\overset{|}{\circ}} - \circ - \circ \cdots$

参考：
通过 W-group 反射得到.
任何有限部分都是单的.

2. Def: Let A be an infinite affine matrix, and $g'(A)$ be the associated Kac–moody algebra, (defined in Rem 1.5 for $n=\infty$).
These Lie algebras are called infinite rank affine algebras.

$$g'(A) = g(A) \quad \text{iff} \quad \det A \neq 0.$$

3. For these Kac-moody algebras we have $\Delta = \Delta^{re}$.
Given $\alpha \in \Delta_+$, denote by $\alpha^\vee (\in H)$, s.t. $\alpha^\vee = [g_\alpha, g_{-\alpha}]$ and $\langle \alpha, \alpha^\vee \rangle = 2$.
put $\Delta^\vee = \{\alpha^\vee \in H' \mid \alpha \in \Delta\}$ to the set of dual roots.

- Denote by gl_∞ the Lie algebra of all complex matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ s.t. the number of nonzero a_{ij} is finite, with the usual braket.

- Let $E_{ij} \in gl_\infty$, $v_i \in \mathbb{C}^\infty$, so that $E_{ij}(v_j) = v_i$.

- Let $A = A_\infty$. Then $g'(A) = sl_\infty := \{a \in gl_\infty \mid \operatorname{tr} a = 0\}$.
  The chevalley generators of $g'(A)$ are: $e_i = E_{i,i+1}$, $f_i = E_{i+1,i}$
  The simple coroots:
  $$\Pi^\vee = \{\alpha_i^\vee = E_{ii} - E_{i+1,i+1} \ (i \in \mathbb{Z})\}.$$

  It is easy to check:
  $$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \quad i,j \in \mathbb{Z}.$$
  $$[h, e_i] = \langle h, e_i \rangle e_i, \quad [h, f_j] = -\langle h, \alpha_i \rangle f_i, \quad h \in H'.$$

Q: $g'(A)$ is a simple Lie algebra? ✓ 取一个 ideal, 取一个非0元. 落在某个单李the中.

- Denotes by ε_i the linear function on H', s.t. $\varepsilon_i(E_{jj}) = \delta_{ij}$ $(j \in \mathbb{Z})$.

  The root system and root spaces of $g'(A)$ are:
  $$\Delta = \{\varepsilon_i - \varepsilon_j \ (i \neq j,\ i,j \in \mathbb{Z})\}, \qquad g_{\varepsilon_i - \varepsilon_j} = \mathbb{C} E_{ij}.$$

  $\varepsilon_i - \varepsilon_j$ being a positive root iff $i < j$.

  The set of simple roots is: $\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \ (i \in \mathbb{Z})\}$.
  $$\langle \alpha_i, \alpha_i^\vee \rangle = \langle E_{ii} - E_{i+1,i+1}, \varepsilon_i - \varepsilon_j \rangle = 2.$$

  The set of positive dual roots is:
  $$\Delta_+^\vee = \underbrace{\Pi^\vee}_{\text{单}} \cup \{\alpha_i^\vee + \alpha_{i+1}^\vee + \cdots + \alpha_j^\vee, \text{ where } i < j,\ i,j \in \mathbb{Z}\}.$$

- The root space decomposition with respect to H' is
  $$g'(A) = H' \oplus \Big(\bigoplus_{i \neq j} g_{\varepsilon_i - \varepsilon_j}\Big).$$

- The triangular decomposition: $g'(A) = n_- \oplus H' \oplus n_+$.